

Volume of Hypercubes Clipped by Hyperplanes and Combinatorial Identities

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Abstract

There was an elegant expression for the volume of hypercube $[0, 1]^n$ clipped by a hyperplane. We generalize the formula to the case of more than one hyperplane. Furthermore we derive several combinatorial identities from the volume expressions of clipped hypercubes.

Keywords: Volume, Hypercube, Combinatorial identity, Optimization

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1. Introduction

A unit hypercube is a convex polytope defined by $[0, 1]^n$ in \mathbb{R}^n . It may seem one of the most basic mathematical objects, but it still has interesting unsolved questions (for examples, see [1]). The unit hypercube itself can be considered as a probabilistic space in many mathematical models describing real phenomena and hence its volume under linear constraints is crucial in statistics and applied mathematics particularly those applications involving probabilistic reasoning like image science [2], machine learning [3] and so on.

It has turned out that volume computation of convex polytopes is algorithmically hard [4] since it usually requires a kind of difficult enumeration like vertex/facet enumeration [5], even for the case of hypercubes clipped by only one hyperplane [6]. For these reasons, both approximation methods and exact calculations have been extensively studied in an algorithmic approach (for example, [7], [8]). This question is not only interesting from an applied point of view, but also there are lots of fascinating mathematics ingeniously involving volume of polytopes; for instance, volume rigidity and polytopal flexibility related with edge lengths [9], Ehrhart theory on counting lattice points [10], and even symplectic geometry [11] and loop quantum gravity [12].

In this paper, we will only focus on closed and concrete formulas from the perspective of elementary linear algebra. Although it doesn't require heavy machinery (but is technically complicated), the resulting volume formula seems to unexpectedly produce a certain class of combinatorial identities. Moreover we expect our elaborated formulation specially focused on $[0, 1]^n$ could be useful to the research direction suggested by P. Filliman (see section 3 in [13]).

For the easiest case of a hypercube clipped by only one hyperplane, there was an interesting simple formula giving the volume as the following. (see section 2 for the notation).

Theorem 1.

$$\text{vol}([0, 1]^n \cap H_1^+) = \sum_{\mathbf{v} \in F^0 \cap H_1^+} \frac{(-1)^{|\mathbf{v}_0|} g_1(\mathbf{v})^n}{n! \prod_{t=1}^n a_t},$$

where the half space H_1^+ is given by

$$\{\mathbf{x} \mid g_1(\mathbf{x}) := \mathbf{a} \cdot \mathbf{x} + r_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n + r_1 \geq 0\}$$

with $\prod_{t=1}^n a_t \neq 0$.

This formula seems to have first appeared in [14], but very similar idea seems to go back much earlier [15]. Although it has been revisited several times (for examples, see section 2 in [16]), a formula for the case of more than one hyperplane had not been discovered yet as far as the authors know. We generalize this formula to the case of an arbitrary number of hyperplanes under the good properties of Theorem 1, in particular, where the expression is written directly in terms of linear coefficients of hyperplanes.

Actually, our volume formulas can be seen as a variant of J. Lawrence's formula [17]. However the expression is more explicit and, in particular, has some benefits for the case of sparse hyperplanes relative to large dimension. Needless to say, one can use our formula even for the case of sufficiently many hyperplanes making up a fully general polytope. But the greater the number of hyperplanes, the less useful the formula seems to be, because the characteristics coming from a hypercube tend to disappear and the formula become the same with Lawrence's one.

General formulas will be presented in Section 4.3. Let us see the case of two hyperplanes beforehand, which is a corollary of Theorem 12 (the detailed description is given in Section 5.1).

Corollary 2.

$$\begin{aligned} \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+) &= \sum_{\mathbf{v} \in F^0 \cap H_1^+ \cap H_2^+} \frac{(-1)^{|\mathbf{v}_0|} g_2(\mathbf{v})^n}{n! \prod_{t=1}^n b_t} \\ &- \sum_{\mathbf{v} \in F^1 \cap H_1 \cap H_2^+} \frac{(-1)^{|\mathbf{v}_0|} a_{*(v)}^n g_2(\mathbf{v})^n}{n! |a_{*(v)}| b_{*(v)} \prod_{t \in [n] \setminus *(v)} \left| \begin{smallmatrix} a_{*(v)} & b_{*(v)} \\ a_t & b_t \end{smallmatrix} \right|}. \end{aligned}$$

where the half spaces H_1^+ and H_2^+ are given by

$$\begin{aligned} H_1^+ &= \{\mathbf{x} \mid g_1(\mathbf{x}) := a_1x_1 + a_2x_2 + \cdots + a_nx_n + r_1 \geq 0\} \\ H_2^+ &= \{\mathbf{x} \mid g_2(\mathbf{x}) := b_1x_1 + b_2x_2 + \cdots + b_nx_n + r_2 \geq 0\} \end{aligned}$$

with good clipping conditions. (See section 2 for the notation and section 4.2 for good clipping conditions.)

Interestingly, volumes of hypercubes clipped by various choices of hyperplanes produce nontrivial combinatorial identities. Let us see several examples.

Theorem 3. For arbitrary $y \in \mathbb{R}$, $a_1, a_2, \dots, a_n \in \mathbb{R}$ and an integer $n \geq 0$,

$$y^n + \sum_{i=1}^n \sum_{1 \leq t_1 < t_2 < \cdots < t_i \leq n} (-1)^i (y + a_{t_1} + \cdots + a_{t_i})^n = (-1)^n n! a_1 a_2 \cdots a_n$$

or equivalently,

$$\sum_{i=1}^n \sum_{1 \leq t_1 < t_2 < \dots < t_i \leq n} (-1)^i (a_{t_1} + \dots + a_{t_i})^k = \begin{cases} -1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, \dots, n-1 \\ (-1)^n n! a_1 a_2 \dots a_n & \text{if } k = n \end{cases}.$$

The identities in the above Theorem 3 can be proved by Theorem 1, but it is related to an old Prouhet-Tarry-Escott problem [18] and algebraic proof is not so difficult. Let us see the following one.

Theorem 4. For arbitrary $y \in \mathbb{R}$, distinct nonzero $a_1, a_2, \dots, a_n \in \mathbb{R}$ and an integer $n \geq 0$,

$$\frac{y^n}{a_1 a_2 \dots a_n} - \sum_{i=1}^n \frac{(y + a_i)^n}{a_i (a_1 - a_i)(a_2 - a_i) \dots \widehat{(a_i - a_i)} \dots (a_n - a_i)} = (-1)^n.$$

or equivalently

$$\sum_{i=1}^n \frac{a_i^k}{(a_1 - a_i)(a_2 - a_i) \dots \widehat{(a_i - a_i)} \dots (a_n - a_i)} = \begin{cases} \frac{1}{a_1 a_2 \dots a_n} & \text{if } k = -1 \\ 0 & \text{if } k = 0, 1, \dots, n-2 \\ (-1)^{n-1} & \text{if } k = n-1 \end{cases}.$$

where $\widehat{}$ means omitting the term.

The above Theorem 4 may be obvious to someone familiar with Vandermonde matrices or Lagrange's interpolation formula. It can be also proved using a volume formula; a special case of Corollary 2 or clipped simplex volume [23]. Furthermore, if we take $a_1 = a_2 = \dots = a_n = 1$ then we get the following corollary which is an extensively studied form in combinatorial enumeration.

Corollary 5. For arbitrary $y \in \mathbb{R}$ and an integer $n \geq 0$.

$$\sum_{i=0}^n (-1)^i \binom{n}{i} (y + i)^n = (-1)^n n!,$$

or equivalently

$$\sum_{i=0}^n (-1)^i \binom{n}{i} i^k = \begin{cases} 0 & \text{if } k = 0, 1, \dots, n-1 \\ (-1)^n n! & \text{if } k = n \end{cases}.$$

Interestingly, the above identities can be unified under one umbrella via a volume expression for a particular clipped hypercube. Before doing that, we make use of the following set-theoretic notation for the sake of convenience,

$$A := \{a_1, a_2, \dots, a_n\}, \quad \|A\| := \sum_{a \in A} a, \quad A! := \prod_{a \in A} a$$

$$R_A(a) := \prod_{b \in A \setminus a} \frac{b}{b - a}, \quad R_A(I) := \sum_{a \in I} R_A(a)$$

Then Theorem 3 and Theorem 4 are written in an economic way as the following.

$$\sum_{I \subset A} (-1)^{|I|} \|I\|^k = \begin{cases} 0 & \text{if } k = 0, 1, \dots, n-1 \\ (-1)^n n! A! & \text{if } k = n \end{cases}$$

$$\sum_{a \in A} R_A(a) a^k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, \dots, n-1 \\ (-1)^{n-1} A! & \text{if } k = n \end{cases}.$$

We can obtain the following identity which can be derived from corollary 2, $m = 2$ case of the volume formula (for the proof, see Theorem 24).

Theorem 6. For $A = \{a_1, \dots, a_n\}$ and an integer $l = 1, \dots, n$,

$$\begin{aligned} & \sum_{|I| < l} (-1)^{|I|} \|I\|^k + \sum_{|I|=l} (-1)^l \|I\|^k R_A(I) \\ &= \begin{cases} 0 & \text{if } k = 0, 1, \dots, n-1 \\ A! \sum_{i=0}^{l-1} (-1)^{n-i} \binom{n}{i} (l-i)^n & \text{if } k = n \end{cases}, \end{aligned}$$

This might also be previously discovered things, although the authors couldn't find any reference in the literature. But the proofs using the volume of clipped hypercubes may be new. Finally we would like to remark that the above identities are all symmetric functions. In section 6 and the Appendix, we will give several identities, some symmetric and others not.

Let us outline our article. we will introduce notation in Section 2 and review and reorganize Lawrence's method in Section 3. The statements of main theorems and proofs will be given in Section 4. Several concrete examples will be presented with more explicit expressions in Section 5. In the final section 6, we will derive a family of combinatorial identities through the volume of clipped hypercubes.

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2. Notations

In this paper, the letters n and m correspond to the dimension of \mathbb{R}^n and the number of hyperplanes respectively except in the appendix. A single bold letter always denotes a vector in \mathbb{R}^n like $\mathbf{x} = (x_1, \dots, x_n)$ and we abuse notation for column vectors and row vectors if it is not confusing. Let \mathbf{e}_i denote the i -th vector in the standard basis of \mathbb{R}^n .

Let K be the natural CW complex structure of unit hypercube $[0, 1]^n$ in \mathbb{R}^n and K^d denote its d -skeleton. We define the *open d -skeleton* F^d as $K^d \setminus K^{d-1}$. Then,

$$[0, 1]^n = \bigcup_{d=0}^n K^d = \bigcup_{d=0}^n F^d, \quad (1)$$

where the \cup symbol denotes disjoint union. For example, $[0, 1]^2$ consists of four points F^0 , four open intervals F^1 and one open rectangle F^2 .

2.1. Index manipulation

Let $[n]$ be an *ordered set* $\{1, 2, \dots, n\}$ which is an index set for the standard basis of \mathbb{R}^n . We will use ordered sets for indices because the sign of a minor of a matrix is sensitive to the order of indices. Let A_I^J and $(A)_I^J$ denote a minor and a submatrix with indices I and J of a matrix $A = (a_{i,j})$ respectively.

For example, let $I = \{1, 3\}$ and $J = \{2, 4\}$ then

$$(A)_I^J = \begin{pmatrix} a_{1,2} & a_{1,4} \\ a_{3,2} & a_{3,4} \end{pmatrix} \text{ and } A_I^J = \begin{vmatrix} a_{1,2} & a_{1,4} \\ a_{3,2} & a_{3,4} \end{vmatrix} = \det \begin{pmatrix} a_{1,2} & a_{1,4} \\ a_{3,2} & a_{3,4} \end{pmatrix}.$$

Let an ordered set $I = \{i_1, i_2, \dots, i_s\} \subset [n]$. Elementary arithmetic operations with an ordered set and a number are done entrywise, for example $2I - 1 = \{2i_1 - 1, \dots, 2i_s - 1\}$. We call I *well-ordered* if $i_1 < i_2 < \dots < i_s$. We consider two different notions of union operation for ordered sets. One is *ordered union* \cup respecting the order between two well-ordered indices, for instance, for $t \notin I$,

$$I \cup \{t\} := \{i_1, i_2, \dots, t, \dots, i_s\} \text{ when } i_1 < i_2 < \dots < t < \dots < i_s.$$

The other is the *joining union* \vee as concatenation.

$$I \vee \{t\} := \{i_1, i_2, \dots, i_s, t\}.$$

We remark that the joining union is defined no matter whether the constituent sets are well-ordered or not, but ordered union is defined only for well-ordered sets. In general, the result of a joining union is not well-ordered. The result of a joining union might be an ordered multi-set.

We abbreviate a set of one element $\{x\}$ to x omitting the brace symbols, for example, $I \vee \{t\} =: I \vee t$. Let I and J be two ordered sets whose underlying unordered sets are the same. Let $\sigma(I, J)$ denote the parity of the permutation between the two ordered sets I and J consisted of the same elements, for example $\sigma(a \vee b, b \vee a) = -1$.

Let $|\cdot|$ and $\|\cdot\|$ denote the cardinality and the total sum of elements of a set respectively. Remember that $\|\emptyset\|^k = 0^k = 1$ when $k = 0$. For $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in \mathbb{R}^n , we define notation \mathbf{v}_0 , \mathbf{v}_1 , \mathbf{v}_{01} and \mathbf{v}_* which denote ordered sets of indices satisfying the following. Be careful not to confuse \mathbf{v} and v .

$$\begin{aligned} \mathbf{v}_0 &:= \{i \in [n] \mid v_i = 0\} & \mathbf{v}_1 &:= \{i \in [n] \mid v_i = 1\} \\ \mathbf{v}_* &:= \{i \in [n] \mid v_i \neq 0, 1\} & \mathbf{v}_{01} &:= \mathbf{v}_0 \cup \mathbf{v}_1 = [n] - \mathbf{v}_* \end{aligned} \quad (2)$$

In particular, we define functions $*_i : \mathbb{R}^n \rightarrow [n]$ and $\bullet_i : \mathbb{R}^n \rightarrow [n]$ by indicating the i -th entry of \mathbf{v}_* and \mathbf{v}_{01} of increasing order respectively, i.e.

$$\begin{aligned}\mathbf{v}_* &= \{i \in [n] \mid v_i \neq 0, 1\} = \{*_1(\mathbf{v}), *_2(\mathbf{v}), \dots, *_{|\mathbf{v}_*|}(\mathbf{v})\}, \\ \mathbf{v}_{01} &= \{i \in [n] \mid v_i = 0, 1\} = \{\bullet_1(\mathbf{v}), \bullet_2(\mathbf{v}), \dots, \bullet_{|\mathbf{v}_{01}|}(\mathbf{v})\}.\end{aligned}$$

If one can consider a set of only one element like $|\mathbf{v}_*| = 1$ then we can omit the index letter like $*(\mathbf{v}) := *_1(\mathbf{v})$. To help understanding, let us see an example. Let $\mathbf{v} = (0, 1, \frac{1}{3}, 0, 0, \frac{3}{5}, 1, \frac{1}{8}) \in F^3$, then we get

$$\begin{aligned}\mathbf{v}_* &= \{3, 6, 8\}, \quad \mathbf{v}_0 = \{1, 4, 5\}, \quad \mathbf{v}_1 = \{2, 7\}, \\ |\mathbf{v}_*| &= 3, \quad |\mathbf{v}_0| = 3, \quad |\mathbf{v}_1| = 2, \\ \|\mathbf{v}_*\| &= 3 + 6 + 8 = 17, \quad \|\mathbf{v}_0\| = 1 + 4 + 5 = 10, \quad \|\mathbf{v}_1\| = 2 + 7 = 9 \\ *_1(\mathbf{v}) &= 3, \quad *_2(\mathbf{v}) = 6, \quad *_3(\mathbf{v}) = 8, \\ \bullet_1(\mathbf{v}) &= 1, \quad \bullet_2(\mathbf{v}) = 2, \quad \bullet_3(\mathbf{v}) = 4, \quad \bullet_4(\mathbf{v}) = 5, \quad \bullet_5(\mathbf{v}) = 7.\end{aligned}$$

Finally, we remark that the following always holds by definition.

$$\mathbf{v}_0 \cup \mathbf{v}_1 \cup \mathbf{v}_* = [n]$$

2.2. Hyperplane matrices

Throughout the article, hyperplanes and half spaces are given by

$$H_i := \{\mathbf{x} \mid g_i(\mathbf{x}) = 0\} \text{ and } H_i^+ := \{\mathbf{x} \mid g_i(\mathbf{x}) \geq 0\},$$

where the linear coefficients are the following.

$$\begin{aligned}g_1(\mathbf{x}) &:= \mathbf{a}_1 \cdot \mathbf{x} + r_1 &= a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n + r_1 \\ &\vdots \\ g_{m-1}(\mathbf{x}) &:= \mathbf{a}_{m-1} \cdot \mathbf{x} + r_{m-1} &= a_{1,m-1}x_1 + a_{2,m-1}x_2 + \dots + a_{n,m-1}x_n + r_{m-1} \\ g_m(\mathbf{x}) &:= \mathbf{a}_m \cdot \mathbf{x} + r_m &= a_{1,m}x_1 + a_{2,m}x_2 + \dots + a_{n,m}x_n + r_m\end{aligned}$$

These coefficients form an $n \times m$ matrix A as the following.

$$A := (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{m-1}, \mathbf{a}_m) = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k-1} & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k-1} & a_{2,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,k-1} & a_{n,m} \end{pmatrix}.$$

We will specially call the $g_m(\mathbf{x})$ and H_m of the last hyperplane as the *auxiliary function* and the *auxiliary hyperplane* respectively. Let H denote the intersection of all half spaces H_i^+ ,

$$H = \bigcap_{i \in [m]} H_i^+.$$

Let I be a set of indices for several hyperplanes except the auxiliary plane, i.e. $I \subset [m-1]$ and let H_I denote the intersection of hyperplanes in I inside the overall intersection $\bigcap H_i^+$ without H_m , i.e.

$$H_I := \bigcap_{j \in I} H_j \cap \bigcap_{i \in [m-1] \setminus I} H_i^+ \cap (H_m^+ \setminus H_m),$$

Remark that we remove the auxiliary plane H_m from the definition of H_I because we should ignore so-called *degenerate* vertices (see section 3.2). Finally, we always assume there are no *redundant* hyperplanes, i.e. for a clipped hypercube,

$$[0, 1]^n \cap \bigcap_{i \in [m]} H_i^+ \neq [0, 1]^n \cap \bigcap_{i \in [m] \setminus j} H_i^+ \text{ for arbitrary } j \in [m]$$

3. A volume formula for convex polytopes

3.1. A pictorial short review on exact volume computations

Let us briefly review conceptual methods to compute the exact volume of convex polytopes in a pictorial way. The volume of an n -parallelotope and an n -simplex given by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in \mathbb{R}^n are obtained from $|\det(\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_n)|$ and $\frac{1}{n!} |\det(\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_n)|$ respectively. An elementary strategy for computing the volume of a polytope is that the polytope is decomposed into a signed summation of simplices. In fact, many volume computing algorithms rely entirely on the method of decomposition as in figure 3.1. The case (a) is the most obvious decomposition that always exists because of convexity.

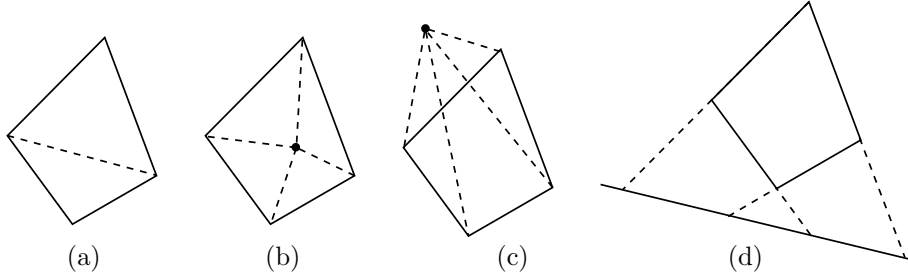


Figure 3.1: Typical decomposition methods for a convex polytope

The cases (b) and (c) are essentially same but for the position of an auxiliary point. These decompositions are quite elementary but interesting because they make an identity between volume of a polytope and volumes of facets (for examples, see J.B. Lasserre [20]).

The case (d) is, in some sense, a dual approach to (c) since it uses an auxiliary plane instead of an auxiliary point. The decomposition of (d) was first invented by J. Lawrence [17]. Let us review the results (for more details, see p.260 in [17]).

Theorem (J. Lawrence).

$$\text{vol}(P) = \sum_{\mathbf{v}: \text{a vertex of } P} N_{\mathbf{v}} \quad \text{with} \quad N_{\mathbf{v}} = \frac{f(\mathbf{v})^n}{n! \delta_{\mathbf{v}} \gamma_1 \gamma_2 \dots \gamma_n}, \quad (3)$$

where f is an auxiliary hyperplane function and the values $\delta_{\mathbf{v}}, \gamma_1 \gamma_2 \cdots \gamma_n$ are obtained by linear algebra calculations (see p.261, 262 in [17] for a precise definition). From a geometric point of view, each $N_{\mathbf{v}}$ is exactly a signed volume of each compact cone which is projected from a vertex \mathbf{v} to the plane $\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0\}$ as in the case (d) in figure 3.1.

In order to make sense of the expression, two conditions are suggested by Lawrence as the following.

Lawrence's two conditions.

- (a) P is a simple polytope, which means that the degrees of vertices in P are same as the dimension of the polytope.
- (b) The auxiliary function f is nonconstant on each edge of P .

To verify the first condition (a), we need to check the number of hyperplanes which meet at each vertex. The second condition (b) is equivalent to the condition that each edge of P is not parallel to the plane $\{f = 0\}$. We remark that (b) is a sufficient condition but not a necessary condition for the volume of each cone not to be ∞ . From this observation, we discuss a slightly different formulation in the next section.

3.2. The volume of a polytope clipped by a hyperplane

Consider a polytope P clipped by a hyperplane $\{f = 0\}$ and apply to it Lawrence's method just as taking f as the auxiliary function.

$$\text{vol} (P \cap \{f \geq 0\}) = \sum_{\substack{\mathbf{v}: \text{a vertex of} \\ P \cap \{f > 0\}}} N_{\mathbf{v}} \quad (4)$$

The formula looks tautologically the same with the previous one, but places emphasis on a subtle point. In this formulation, we can easily observe that there are valid situations which violate Lawrence's two assumptions. Even if a non-simple vertex or a parallel edge in P itself is placed on the auxiliary plane, we don't need to evaluate it and check any condition at all. We would say that a vertex or an edge is *degenerate* if it is contained in the auxiliary plane, or *non-degenerate* otherwise. In this paper, all vertices in volume formulas are always assumed to be non-degenerate or a certain term of a denominator may be zero.

In deriving our formula, we will suggest more explicit assumptions to make the formula valid, which are a little different from Lawrence's conditions. At this stage, we would like to remark that there is a subtle case which satisfies Lawrence's assumptions but is not applicable for our formula. At first glance, it might seem weird because the Lawrence formula and our formula are essentially the same algorithm to compute volume. Although they are conceptually the same, a difference appears during a very concrete formulation. We will discuss this issue in Section 4.2.

3.3. A volume formula of convex polytopes

We rewrite Lawrence's formula in closed form directly in terms of linear coefficients of hyperplanes. In fact, Lawrence's method itself can be considered as an explicit expression in terms of linear coefficients (for example, see p.393 [21]). But it is still not enough to proceed in our formulation, so we will derive it in a more elaborated way. We take an

auxiliary plane as the last hyperplane H_m , not an additional plane, i.e. $P = P \cap H_m^+$. Remember that $m > n$ in order to define a convex compact polytope.

Theorem 7. *Let a convex polyhedron $P = \bigcap_{i \in [m-1]} H_i^+ \cap H_m^+$ with all non-degenerate vertices simple and g_m non-constant on each non-degenerate edge of P . Then the volume is*

$$\text{vol}(P \cap H_m^+) = \sum_{I \subset [m-1]}^{|I|=n} \sum_{\mathbf{v} \in H_I} \frac{(-1)^{\frac{n(n+1)}{2}} (g_m(\mathbf{v}) A_{[n]}^I)^n}{n! |A_{[n]}^I| \prod_{t \in I} A_{[n]}^{I \setminus t \cup m}}.$$

Remember the definition of H_I which makes us exclude all degenerate vertices automatically.

Remark 1. In the formula, the second summation consists of either an empty summand or only one summand. In spite of the redundancy of the expression, we would like to persist in the inefficient form for the main theorem on a clipped hypercube. Note that $I \cup m \setminus t = I \setminus t \cup m = I \setminus t \vee m$ because m is the last element.

Proof. Let us just do a direct computation in our setup from Lawrence's formula of (4). Because of the simple and non-degenerate conditions, each vertex \mathbf{v} is an intersection of exactly n hyperplanes other than the auxiliary plane,

$$H_{i_1}, H_{i_2}, \dots, H_{i_n} \quad (1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq m-1)$$

with $\mathbf{v} = \bigcap_{t=1}^n H_{i_t}$. Let $I := I_{\mathbf{v}} := (i_1, \dots, i_n)$ and then

$$(-A)_{[n]}^I = (-\mathbf{a}_{i_1}, -\mathbf{a}_{i_2}, \dots, -\mathbf{a}_{i_n}).$$

Let $\boldsymbol{\gamma} := \boldsymbol{\gamma}_{\mathbf{v}} := (\gamma_1, \dots, \gamma_n)^t$ where each γ_i is written in (3). It follows from the definition in [17] that $\boldsymbol{\gamma}$ is defined as satisfying

$$\mathbf{a}_m = (-A)_{[n]}^I \boldsymbol{\gamma},$$

So we have $\boldsymbol{\gamma} = ((-A)_{[n]}^I)^{-1} \mathbf{a}_m$.

Let $(x_{ij}) := ((-A)_{[n]}^I)^{-1}$ then we get $x_{ij} = \frac{(-1)^{i+j} ((-A)_{[n]}^I)_{j,i}}{\det((-A)^I)}$ by Cramer's rule where $((-A)_{[n]}^I)_{j,i}$ means the (j, i) -minor of the submatrix $(-A)_{[n]}^I$, i.e. $|(-A)_{[n] \setminus j}^{I \setminus i}|$ with $I = \{I_1, I_2, \dots, I_n\}$. Then

$$\begin{aligned} \boldsymbol{\gamma} &= \begin{pmatrix} \sum_j a_{j,m} x_{1,j} \\ \sum_j a_{j,m} x_{2,j} \\ \vdots \\ \sum_j a_{j,m} x_{n,j} \end{pmatrix} = \frac{1}{\det((-A)^I)} \begin{pmatrix} \sum_j (-1)^{1+j} a_{j,m} ((-A)_{[n]}^I)_{j,1} \\ \sum_j (-1)^{2+j} a_{j,m} ((-A)_{[n]}^I)_{j,2} \\ \vdots \\ \sum_j (-1)^{n+j} a_{j,m} ((-A)_{[n]}^I)_{j,n} \end{pmatrix} \\ &= \frac{1}{\det((-A)^I)} \begin{pmatrix} \det(\mathbf{a}_m, -\mathbf{a}_{i_2}, \dots, -\mathbf{a}_{i_n}) \\ \det(-\mathbf{a}_{i_1}, \mathbf{a}_m, \dots, -\mathbf{a}_{i_n}) \\ \vdots \\ \det(-\mathbf{a}_{i_1}, -\mathbf{a}_{i_2}, \dots, \mathbf{a}_m) \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned}
\gamma_1 \gamma_2 \cdots \gamma_n &= \prod_{j=1}^n \frac{\det(-\mathbf{a}_{i_1}, \dots, -\mathbf{a}_{i_{j-1}}, \mathbf{a}_m, -\mathbf{a}_{i_{j+1}}, \dots, -\mathbf{a}_{i_n})}{\det((-A)_{[n]}^I)} \\
&= \prod_{j=1}^n \frac{(-1)^{n-1} \det(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_{j-1}}, \mathbf{a}_m, \mathbf{a}_{i_{j+1}}, \dots, \mathbf{a}_{i_n})}{(-1)^n \det(A_{[n]}^I)} \\
&= (-1)^n (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^n \frac{\det(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_{j-1}}, \mathbf{a}_{i_{j+1}}, \dots, \mathbf{a}_{i_n}, \mathbf{a}_m)}{\det(A_{[n]}^I)} \\
&= \frac{(-1)^{\frac{n(n+1)}{2}}}{\det(A_{[n]}^I)^n} \prod_{t \in I} A_{[n]}^{I \setminus t \cup m},
\end{aligned}$$

also we get

$$\delta_{\mathbf{v}} = |\det((-A)_{[n]}^I)| = |\det(A_{[n]}^I)| = |A_{[n]}^I|.$$

Therefore we proved the theorem. \square

We take an auxiliary hyperplane g_m among the hyperplanes comprising a polytope in the above theorem. For the case of taking any other auxiliary hyperplane, it suffices to add the normal vector into the last column of A . Then everything works well.

4. Volume formulas for clipped hypercubes

4.1. Hyperplanes and indices for clipped hypercubes

An n -dimensional unit hypercube $[0, 1]^n$ is given by $2n$ half spaces,

$$x_i \geq 0 \quad \text{and} \quad x_i \leq 1 \quad \text{for} \quad i \in [n].$$

Let P be a hypercube clipped by hyperplanes H_1, \dots, H_m , i.e.

$$P := [0, 1]^n \cap \bigcap_{i=1}^m H_i.$$

From now on, we need to distinguish between the hyperplanes \mathcal{H}_i defining P and the hyperplanes H_i which are not hyperplanes of $[0, 1]^n$ in order to apply Theorem 7.

Let \mathcal{H}_i^+ be defined as the followings.

$$\mathcal{H}_i^+ = \begin{cases} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{e}_{(i+1)/2} \cdot \mathbf{x} \geq 0\} & \text{if } i \in [2n] \text{ is odd} \\ \{\mathbf{x} \in \mathbb{R}^n \mid (-\mathbf{e}_{i/2}) \cdot \mathbf{x} + 1 \geq 0\} & \text{if } i \in [2n] \text{ is even} \\ H_{i-2n}^+ & \text{if } i \in [2n+m] \setminus [2n]. \end{cases} \quad (5)$$

Similarly we consider a big coefficient matrix \mathcal{A} and indices \mathcal{I} for \mathcal{H}_i from a given hyperplane matrix A and indices I for H_i , respectively. Let \mathcal{A} be an $n \times (2n+m)$ matrix as follows.

$$\mathcal{A} = (\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2, \dots, \mathbf{e}_n, -\mathbf{e}_n \mid A)$$

$$= \begin{pmatrix} 1 & -1 & & & a_{11} & a_{1,m-1} & a_{1,m} \\ & 1 & -1 & & a_{21} & a_{2,m-1} & a_{2,m} \\ & & \ddots & & \vdots & \dots & \vdots \\ & & & 1 & -1 & a_{n1} & a_{n,m-1} & a_{n,m} \end{pmatrix} \quad (6)$$

For a simple vertex \mathbf{v} , there is an index set \mathcal{I} indicating n hyperplanes $\mathcal{H}_{i_1}, \mathcal{H}_{i_2}, \dots, \mathcal{H}_{i_n}$ making up the vertex \mathbf{v} , i.e.

$$\mathbf{v} = \bigcap_{k \in \mathcal{I}} \mathcal{H}_k \quad \text{for } \mathcal{I} = \mathcal{I}(\mathbf{v}) = \{i_1, \dots, i_n\} \subset [2n + m],$$

where \mathcal{I} is well-ordered, $i_1 < i_2 < \dots < i_n$. We decompose \mathcal{I} into two parts,

$$\mathcal{I} = \mathcal{I}_{01} \cup \mathcal{I}_* \quad (7)$$

with $\mathcal{I}_{01} = \{i \in \mathcal{I} | i \in [2n]\}$ and $\mathcal{I}_* = \{i \in \mathcal{I} | i \in [2n + m] \setminus [2n]\}$. We can check that

$$(\mathcal{A})_{[n]}^{\mathcal{I}_*} = (\mathcal{A})_{[n]}^{\mathcal{I}}. \quad (8)$$

The following obvious lemma says that each vertex has a natural grading from F^d . Our volume formulas can be considered as a summation over this grading.

Lemma 8. *For a simple vertex \mathbf{v} of a clipped hypercube P , there is an index set $I \subset [n]$ such that $\mathbf{v} \in F^d \cap H_I$. Moreover, $|\mathbf{v}_*| = |I| = d$.*

Proof. Let $\mathbf{v} = (v_1, v_2, \dots, v_n) \in F^d$, then $|\mathbf{v}_{01}| = |\mathcal{I}_{01}| = n - d$ because $v_i = 0$ or 1 if and only if \mathbf{v} intersects a hyperplane of $x_i = 0$ or $x_i = 1$. Since $\mathcal{I}_* = I + 2n$ and $|\mathcal{I}_*| = |I|$, the lemma is trivial by (2) and (7). \square

Let $\mathbf{v} \in F^d$ and see this in more detail,

$$\mathbf{v} \in \bigcap_{i \in \mathbf{v}_0} \mathcal{H}_{2i-1} \cap \bigcap_{i \in \mathbf{v}_1} \mathcal{H}_{2i}. \quad (9)$$

Let $\mathcal{I}_0 := \{i \in \mathcal{I}_{01} \mid i \text{ is odd}\}$ and $\mathcal{I}_1 := \{i \in \mathcal{I}_{01} \mid i \text{ is even}\}$, then it immediately follows that

$$\mathcal{I}_0 = (2\mathbf{v}_0 - 1) \text{ and } \mathcal{I}_1 = 2\mathbf{v}_1. \quad (10)$$

Finally we also obtain the following obvious lemma.

Lemma 9. $(\mathcal{A})_{\mathbf{v}_{01}}^{\mathcal{I}_{01}}$ is a diagonal matrix with

$$((\mathcal{A})_{\mathbf{v}_{01}}^{\mathcal{I}_{01}})_{i,i} = \begin{cases} 1 & \text{if } v_{\bullet(i)} = 0 \\ -1 & \text{if } v_{\bullet(i)} = 1, \end{cases}$$

in particular, $\mathcal{A}_{\mathbf{v}_{01}}^{\mathcal{I}_{01}} = (-1)^{|\mathbf{v}_1|}$.

Proof. The hyperplanes \mathcal{H}_{2i} and \mathcal{H}_{2i-1} are parallel and never intersect. Hence, for any $t \in [n]$, $2t$ and $2t - 1$ cannot be contained in \mathcal{I}_{01} at the same time. By the definition of \mathcal{A} and (10), the matrix is diagonal with $|\mathbf{v}_0|$ number of 1 entries and $|\mathbf{v}_1|$ number of -1 entries. \square

4.2. Good clipping conditions

We discuss two conditions to make sense of our volume formula, which also may be considered as an explicit form of Lawrence's two conditions in Section 3.1 but with slightly different meaning. For example, as mentioned in Section 3.2, we don't need to check any assumption for nondegenerate vertices or edges. So there are valid cases even for non-simple polytopes. The following proposition gives a sufficient condition to justify our formula.

Proposition 10 (Good clipping conditions). *If a clipped hypercube given by Section 4.1 satisfies the following assumptions,*

- (A) *For any $I \subset [m-1]$, $F^{|I|-1} \cap H_I = \emptyset$.*
- (B) *For any $I \subset [m-1]$ and $\mathbf{v} \in F^{|I|} \cap H_I$, $\prod_{t \in I} A_{\mathbf{v}_*}^{I \cup m \setminus t} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \cup t}^{I \cup m} \neq 0$.*

then Lawrence's two conditions hold for non-degenerate vertices.

Proof. Let us see that (A) implies that every non-degenerate vertex is simple. If a non-simple vertex \mathbf{v} exists then there are at least $n+1$ hyperplanes intersecting \mathbf{v} , which meet k hyperplanes of $[0, 1]^n$ and $H_{i_1}, \dots, H_{i_{n-k}}$ for $0 \leq k \leq n+1$. Then $\mathbf{v} \in F^{n-k} \cap H_I$ with $I = \{i_1, \dots, i_{n-k}\}$, this conflicts with (A). The condition of (B) means the volume of the cone at \mathbf{v} is finite, which is equivalent to Lawrence's assumption (b) for non-degenerate vertices. \square

We mention that the above good clipping assumptions give no restrictions about degenerate vertices. We should remark that the converse of Proposition 10 is not true as the following shows.

Example 1. There is a clipped hypercube which is simple but violates good clipping condition (A). Consider $P = [0, 1]^3 \cap H_1^+ \cap H_2^+$ where

$$\begin{aligned} H_1^+ &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + 2x_3 \geq 1\}, \\ H_2^+ &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 \leq 1\}, \end{aligned}$$

We can see that all vertices are simple, but

$$\begin{aligned} H_1 \cap F^{1-1} &= \{(1, 0, 0), (0, 1, 0)\} \neq \emptyset \\ H_2 \cap F^{1-1} &= \{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\} \neq \emptyset. \end{aligned}$$

The reason this situation occurs is that we always take all $2n$ hyperplanes making up $[0, 1]^n$ when we compute a volume expression. So there can be a simple vertex in a clipped hypercube which is the intersection of more than $n+1$ hyperplanes \mathcal{H}_i for $i \in [2n+m]$.

Remark 2. Assumption (A) and (B) are generic conditions which means that the conditions always hold for polytopes in general position. In other words, we can always find a clipped hypercube satisfying good clipping conditions which has only an arbitrarily small difference from the original polytope.

When we want to apply Lawrence's formula to non-simple polytopes, we could imagine a decomposition at a non-simple vertex into simple cones using a "lexicographic rule" ([17], [21]). However, by the observation of the remark above, we suggest another method, called ϵ -perturbation, for the cases violating good clipping conditions. We add a perturbation variable ϵ into a proper position of datum of hyperplanes so as that the perturbed polytopes P_ϵ are always simple for sufficient small $\epsilon > 0$ then we can write the volume $\text{vol}(P_\epsilon)$ using the explicit formula and take the limit as $\epsilon \rightarrow 0$. We give several examples for ϵ -perturbation in Section 6 and the Appendix. We will discuss a more detailed recipe of specific ϵ -perturbation in a follow-up article [22]. The strategy is nothing special but it is usually difficult to compute a precise limit. We need to formulate sufficiently concrete expressions in Section 5 to compute exact limits.

4.3. Several volume formulas

We present our main theorems. See Section 2 and 4.1 for notations.

Theorem 11. *The volume of a hypercube clipped by m halfspaces H_1, H_2, \dots, H_m satisfying good clipping conditions in Section 4.2 is given by*

$$\text{vol}([0, 1]^n \cap H) = \sum_{I \subset [m-1]} \sum_{\mathbf{v} \in F^{|I|} \cap H_I} \frac{(-1)^{|\mathbf{v}_0| + \|\mathbf{v}_*\|} (g_m(\mathbf{v}) A_{\mathbf{v}_*}^I)^n}{n! |A_{\mathbf{v}_*}^I| \prod_{t \in I} A_{\mathbf{v}_*}^{I \cup m \setminus t} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \cup t}^{I \cup m}}.$$

There is another expression in terms of joining union \vee instead of ordered union \cup as follows. The only difference between them is $\frac{|I|(|I|+1)}{2}$ and $\|\mathbf{v}_*\|$.

Theorem 12. *Under the same hypotheses as Theorem 11,*

$$\text{vol}([0, 1]^n \cap H) = \sum_{I \subset [m-1]} \sum_{\mathbf{v} \in F^{|I|} \cap H_I} \frac{(-1)^{|\mathbf{v}_0| + \frac{|I|(|I|+1)}{2}} (g_m(\mathbf{v}) A_{\mathbf{v}_*}^I)^n}{n! |A_{\mathbf{v}_*}^I| \prod_{t \in I} A_{\mathbf{v}_*}^{I \vee m \setminus t} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I \vee m}}.$$

The following proposition implies that the above two theorems are equivalent.

Proposition 13. *Under the same hypotheses as Theorem 11,*

$$\prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \cup t}^{I \cup m} = (-1)^{\|\mathbf{v}_*\| - \frac{|I|(|I|+1)}{2}} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I \vee m}.$$

Proof. The set $\mathbf{v}_{01} = [n] \setminus \mathbf{v}_*$ is divided into $|I| + 1$ (possibly empty) blocks,

$$\begin{aligned} & \{1, 2, \dots, i_1 - 1\}, \\ & \{i_1 + 1, i_1 + 2, \dots, i_2 - 1\}, \\ & \vdots \\ & \{i_{|I|} + 1, i_{|I|} + 2, \dots, n\}. \end{aligned}$$

Whenever $t \in \mathbf{v}_{01}$ is placed in each block, it requires $|I|, |I| - 1, \dots, 1, 0$ transpositions. Hence the total number of transpositions is

$$(i_1 - 1)|I| + (i_2 - i_1 - 1)(|I| - 1) + \dots + (i_{|I|} - i_{|I|-1} - 1)1 + (n - i_{|I|})0$$

$$\begin{aligned}
&= i_1 + i_2 \cdots + i_{|I|} - (|I| + (|I| - 1) + \cdots + 1) \\
&= \|\mathbf{v}_*\| - \frac{|I|(|I| + 1)}{2}.
\end{aligned}$$

□

Note that ordered union is commutative but joining union is not, i.e. $I \cup m = m \cup I$ but $I \vee m \neq m \vee I$. Thus, in Theorem 11 the expression order of union operations doesn't matter. But if one takes joining union as in Theorem 12, there are several choices of expressions between $m \vee I$ and $I \vee m$ because it is sensitive to changing order. The following lemma shows that orders of the expressions affects few things in Theorem 12.

Lemma 14. *Under the same hypotheses as Theorem 11,*

$$\begin{aligned}
\prod_{t \in I} A_{\mathbf{v}_*}^{I \vee m \setminus t} &= \prod_{t \in I} A_{\mathbf{v}_*}^{m \vee I \setminus t} = \prod_{t \in I} A_{\mathbf{v}_*}^{I \cup m \setminus t} = \prod_{t \in I} A_{\mathbf{v}_*}^{m \cup I \setminus t} \\
\prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I \vee m} &= \prod_{t \in \mathbf{v}_{01}} A_{t \vee \mathbf{v}_*}^{m \vee I} = (-1)^{|I|(n-|I|)} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{m \vee I} = (-1)^{|I|(n-|I|)} \prod_{t \in \mathbf{v}_{01}} A_{t \vee \mathbf{v}_*}^{I \vee m}.
\end{aligned}$$

Proof. This is obvious by Lemma 8 along with

$$\sigma(m \cup I, I \cup m) = 1 \quad \text{and} \quad \sigma(m \vee I, I \vee m) = (-1)^{|I|}.$$

□

Therefore it may be sufficient to mention the following version of joining union which is almost the same as Theorem 12. The only difference is a sign change from $\frac{|I|(|I|+1)}{2}$ to $\frac{|I|(|I|-1)}{2} + n|I|$

Theorem 15. *Under the same hypotheses as Theorem 11,*

$$\text{vol}([0, 1]^n \cap H) = \sum_{I \subset [m-1]} \sum_{\mathbf{v} \in F^{|I|} \cap H_I} \frac{(-1)^{|\mathbf{v}_0| + \frac{|I|(|I|-1)}{2} + n|I|} (g_m(\mathbf{v}) A_{\mathbf{v}_*}^I)^n}{n! |A_{\mathbf{v}_*}^I| \prod_{t \in I} A_{\mathbf{v}_*}^{I \vee m \setminus t} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{m \vee I}}.$$

4.4. Separating parity

We introduce a *separating parity* $\Delta(I, J)$ of two indices $I \supset J$ for effective bookkeeping of complicated permutation parity.

Definition 16.

$$\Delta(I, J) := \sigma(I, (I \setminus J) \vee J).$$

We need the following lemmas for convenience.

Lemma 17. *For any $I \subseteq [n]$,*

$$\Delta([n], I) = (-1)^{n|I| - \|I\| - \frac{|I|(|I|-1)}{2}},$$

in particular, for $t \in [n]$

$$\Delta([n], t) = (-1)^{n-t}.$$

Proof. Let $I = \{i_1, \dots, i_{|I|}\}$, then

$$\begin{aligned} [n] \setminus I &= \{1, \dots, i_1 - 1, \widehat{i_1}, i_1 + 1, \dots, i_{|I|} - 1, \widehat{i_{|I|}}, i_{|I|} + 1, \dots, n\} \\ ([n] \setminus I) \vee I &= \{1, \dots, i_1 - 1, i_1 + 1, \dots, i_{|I|} - 1, i_{|I|} + 1, \dots, n\} \vee \{i_1, \dots, i_{|I|}\}. \end{aligned}$$

We just count the number of transpositions. In order to shift each i_t in $([n] \setminus I) \vee I$ into its original position in $[n]$, it requires $n - i_t - (|I| - t)$ transpositions.

$$\sum_{t \in [|I|]} n - i_t - (|I| - t) = n|I| - \|I\| - \frac{|I|(|I| - 1)}{2}. \quad \square$$

Lemma 18. For any $I \subseteq [n]$,

$$\prod_{i \in I} \Delta(I, i) = (-1)^{\frac{|I|(|I|-1)}{2}}.$$

Proof. Let $I = \{i_1, i_1, \dots, i_{|I|}\}$. Each $i_t = i$ requires $|I| - t$ transposition. Hence,

$$(-1)^{|I|-1} (-1)^{|I|-2} \dots (-1)^{|I|-|I|} = (-1)^{\frac{|I|(|I|-1)}{2}}. \quad \square$$

4.5. Proof of the volume formula for clipped hypercubes

Let us prove Theorem 12. At first, the summation over \mathcal{I} and $\mathcal{H}_{\mathcal{I}}$ applied to Theorem 7 is converted into a summation over I and H_I as follows.

$$\begin{aligned} \sum_{\substack{|\mathcal{I}|=n \\ \mathcal{I} \subset [2n+m-1]}} \sum_{\mathbf{v} \in \mathcal{H}_{\mathcal{I}}} &= \sum_{\substack{|\mathcal{I}_{01} \vee \mathcal{I}_*|=n \\ \mathcal{I}_{01} \vee \mathcal{I}_* \subset [2n+m-1]}} \sum_{\mathbf{v} \in \mathcal{H}_{\mathcal{I}_{01}} \cap \mathcal{H}_{\mathcal{I}_*}} \\ &= \sum_{\substack{\mathcal{I}_* \subset [2n+m-1] \setminus [2n] \\ |\mathcal{I}_{01}| + |\mathcal{I}_*| = n}} \sum_{\mathcal{I}_{01} \subset [2n]} \sum_{\mathbf{v} \in \mathcal{H}_{\mathcal{I}_{01}} \cap \mathcal{H}_{\mathcal{I}_*}} \\ &= \sum_{\substack{\mathcal{I}_* \subset [m-1] + 2n \\ |\mathcal{I}_*| + |\mathcal{I}_*| = n}} \sum_{\mathbf{v} \in F^{|\mathcal{I}_*|} \cap \mathcal{H}_{\mathcal{I}_*}} \\ &= \sum_{I \subset [m-1]} \sum_{\mathbf{v} \in F^{|I|} \cap H_I} \end{aligned}$$

For convenience sake, Δ denotes the separating parity of $[n]$ and \mathbf{v}_* , i.e.

$$\Delta := \Delta([n], \mathbf{v}_*) = \sigma(\mathbf{v}_{01} \cup \mathbf{v}_*, \mathbf{v}_{01} \vee \mathbf{v}_*) \quad (11)$$

Then, we derive several relations between minors of A and \mathcal{A} .

Proposition 19.

$$\mathcal{A}_{[n]}^{\mathcal{I}} = (-1)^{|\mathbf{v}_1|} \Delta A_{\mathbf{v}_*}^I$$

Proof. By Lemma 9 and (8),

$$\begin{aligned} \mathcal{A}_{[n]}^{\mathcal{I}} &= \mathcal{A}_{\mathbf{v}_{01} \cup \mathbf{v}_*}^{\mathcal{I}_{01} \vee \mathcal{I}_*} = \Delta \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{\mathcal{I}_{01} \vee \mathcal{I}_*} \\ &= \Delta \mathcal{A}_{\mathbf{v}_{01}}^{\mathcal{I}_{01}} \mathcal{A}_{\mathbf{v}_*}^{\mathcal{I}_*} = (-1)^{|\mathbf{v}_1|} \Delta A_{\mathbf{v}_*}^I. \end{aligned} \quad \square$$

Recall Lemma 9 and (8); the matrix $\mathcal{A}_{[n]}^{\mathcal{I}_{01}}$ is diagonal and $\mathcal{A}_{[n]}^{\mathcal{I}_*} = A_{[n]}^I$. Hence we need to decompose $\mathcal{A}_{[n]}^{\mathcal{I} \setminus t \cup m}$ into \mathcal{I}_{01} and \mathcal{I}_* as the following.

$$\prod_{t \in \mathcal{I}} \mathcal{A}_{[n]}^{\mathcal{I} \setminus t \cup m} = \prod_{t \in \mathcal{I}_{01}} \mathcal{A}_{[n]}^{(\mathcal{I}_{01} \setminus t) \cup \mathcal{I}_* \cup m} \prod_{t \in \mathcal{I}_*} \mathcal{A}_{[n]}^{\mathcal{I}_{01} \cup (\mathcal{I}_* \setminus t) \cup m}. \quad (12)$$

First, the case of $\mathcal{I}_* \setminus t$ is proved like Proposition 19.

Proposition 20.

$$\prod_{t \in \mathcal{I}_*} \mathcal{A}_{[n]}^{\mathcal{I}_{01} \vee (\mathcal{I}_* \setminus t \vee m)} = (-1)^{|I||\mathbf{v}_1|} \Delta^{|I|} \prod_{t \in I} A_{\mathbf{v}_*}^{I \setminus t \vee m}$$

The case $\mathcal{I}_{01} \setminus t$ is much more complicated than the previous case.

Proposition 21.

$$\prod_{t \in \mathcal{I}_{01}} \mathcal{A}_{[n]}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} = (-1)^{|\mathbf{v}_*||\mathbf{v}_0| + |\mathbf{v}_1||\mathbf{v}_0| + |\mathbf{v}_*||\mathbf{v}_1| + \frac{|\mathbf{v}_{01}|(|\mathbf{v}_{01}|-1)}{2}} \Delta^{|\mathbf{v}_{01}|} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I_* \vee m}$$

Proof. First, divide \mathcal{I}_{01} into \mathcal{I}_0 and \mathcal{I}_1 , which correspond to \mathbf{v}_0 and \mathbf{v}_1 respectively.

$$\begin{aligned} \prod_{t \in \mathcal{I}_{01}} \mathcal{A}_{[n]}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} &= \prod_{t \in \mathcal{I}_{01}} \mathcal{A}_{\mathbf{v}_{01} \cup \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} \\ &= \prod_{t \in \mathcal{I}_{01}} \Delta \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} \\ &= \Delta^{|\mathcal{I}_{01}|} \prod_{t \in \mathcal{I}_0} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} \prod_{t \in \mathcal{I}_1} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} \\ &= \Delta^{|\mathcal{I}_{01}|} \prod_{t \in \mathbf{v}_0} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus (2t-1)) \vee \mathcal{I}_* \vee m} \prod_{t \in \mathbf{v}_1} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus 2t) \vee \mathcal{I}_* \vee m}. \end{aligned}$$

Each term is computed as follows.

$$\begin{aligned} \prod_{t \in \mathbf{v}_0} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus (2t-1)) \vee \mathcal{I}_* \vee m} &= \prod_{t \in \mathbf{v}_0} \Delta(\mathbf{v}_{01}, t) \mathcal{A}_{(\mathbf{v}_{01} \setminus t) \vee t \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus (2t-1)) \vee \mathcal{I}_* \vee m} \\ &= (-1)^{|\mathbf{v}_*||\mathbf{v}_0|} \prod_{t \in \mathbf{v}_0} \Delta(\mathbf{v}_{01}, t) \mathcal{A}_{(\mathbf{v}_{01} \setminus t) \vee \mathbf{v}_* \vee t}^{(\mathcal{I}_{01} \setminus (2t-1)) \vee \mathcal{I}_* \vee m} \\ &= (-1)^{|\mathbf{v}_*||\mathbf{v}_0|} \prod_{t \in \mathbf{v}_0} \Delta(\mathbf{v}_{01}, t) \mathcal{A}_{\mathbf{v}_{01} \setminus t}^{\mathcal{I}_{01} \setminus (2t-1)} A_{\mathbf{v}_* \vee t}^{\mathcal{I}_* \vee m} \end{aligned}$$

$$\begin{aligned} \text{(by Lemma 9, consider } \mathcal{A}_{\mathbf{v}_{01}}^{\mathcal{I}_{01}} &= \begin{cases} \mathcal{A}_{\mathbf{v}_{01} \setminus t}^{\mathcal{I}_{01} \setminus (2t-1)} & \text{if } t \in \mathbf{v}_0 \\ -\mathcal{A}_{\mathbf{v}_{01} \setminus t}^{\mathcal{I}_{01} \setminus 2t} & \text{if } t \in \mathbf{v}_1 \end{cases}) \\ &= (-1)^{|\mathbf{v}_*||\mathbf{v}_0|} \prod_{t \in \mathbf{v}_0} \Delta(\mathbf{v}_{01}, t) (-1)^{|\mathbf{v}_1|} \mathcal{A}_{\mathbf{v}_* \vee t}^{\mathcal{I}_* \vee m} \\ &= (-1)^{(|\mathbf{v}_*| + |\mathbf{v}_1|)|\mathbf{v}_0|} \prod_{t \in \mathbf{v}_0} \Delta(\mathbf{v}_{01}, t) \mathcal{A}_{\mathbf{v}_* \vee t}^{I_* \vee m}. \end{aligned} \quad (13)$$

Similarly,

$$\prod_{t \in \mathbf{v}_1} \mathcal{A}_{\mathbf{v}_{01} \vee \mathbf{v}_*}^{(\mathcal{I}_{01} \setminus 2t) \vee \mathcal{I}_* \vee m} = (-1)^{(|\mathbf{v}_*| + |\mathbf{v}_1|)|\mathbf{v}_1| - |\mathbf{v}_1|} \prod_{t \in \mathbf{v}_1} \Delta(\mathbf{v}_{01}, t) A_{\mathbf{v}_* \vee t}^{I_* \vee m}. \quad (14)$$

Take (13) and (14) together to complete the proof.

$$\begin{aligned} \prod_{t \in \mathcal{I}_{01}} \mathcal{A}_{[n]}^{(\mathcal{I}_{01} \setminus t) \vee \mathcal{I}_* \vee m} &= (-1)^{(|\mathbf{v}_*| + |\mathbf{v}_1|)(|\mathbf{v}_0| + |\mathbf{v}_1|) - |\mathbf{v}_1|} \Delta^{|\mathcal{I}_{01}|} \prod_{t \in \mathbf{v}_{01}} \Delta(\mathbf{v}_{01}, t) A_{\mathbf{v}_* \vee t}^{I_* \vee m} \\ &= (-1)^{|\mathbf{v}_*||\mathbf{v}_0| + |\mathbf{v}_1||\mathbf{v}_0| + |\mathbf{v}_*||\mathbf{v}_1| + \frac{|\mathbf{v}_{01}|(|\mathbf{v}_{01}| - 1)}{2}} \Delta^{|\mathbf{v}_{01}|} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I_* \vee m}. \quad \square \end{aligned}$$

We put the three propositions 19, 20 and 21 together.

$$\begin{aligned} &\sum_{\mathcal{I} \subset [m-1]}^{|I|=n} \sum_{\mathbf{v} \in \mathcal{H}_{\mathcal{I}}} \frac{(-1)^{\frac{n(n+1)}{2}} (g_m(\mathbf{v}) \mathcal{A}_{[n]}^{\mathcal{I}})^n}{n! |\mathcal{A}_{[n]}^{\mathcal{I}}| \prod_{t \in \mathcal{I}} \mathcal{A}_{[n]}^{T \cup m}} \\ &= \sum_{I \subset [m-1]} \sum_{\mathbf{v} \in F^{I \setminus I} \cap H_I} \frac{(-1)^{\frac{n(n+1)}{2} + n|\mathbf{v}_1| + |I||\mathbf{v}_1| + |\mathbf{v}_*||\mathbf{v}_0| + |\mathbf{v}_1||\mathbf{v}_0| + |\mathbf{v}_*||\mathbf{v}_1| + \frac{|\mathbf{v}_{01}|(|\mathbf{v}_{01}| - 1)}{2}} \Delta^n (g_m(\mathbf{v}) A_{\mathbf{v}_*}^I)^n}{n! |A_{\mathbf{v}_*}^I| \Delta^{|I| + |\mathbf{v}_{01}|} \prod_{t \in I} A_{\mathbf{v}_*}^{I \setminus t \vee m} \prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I_* \vee m}}. \end{aligned}$$

We calculate the parity expression,

$$\begin{aligned} &\frac{n(n+1)}{2} + n|\mathbf{v}_1| + |I||\mathbf{v}_1| + |\mathbf{v}_*||\mathbf{v}_0| + |\mathbf{v}_1||\mathbf{v}_0| + |\mathbf{v}_*||\mathbf{v}_1| + \frac{|\mathbf{v}_{01}|(|\mathbf{v}_{01}| - 1)}{2} \\ &\equiv_{(\text{mod } 2)} \frac{n(n+1)}{2} + \frac{|\mathbf{v}_{01}|(|\mathbf{v}_{01}| + 1)}{2} - |\mathbf{v}_{01}| + n|\mathbf{v}_1| + |\mathbf{v}_*||\mathbf{v}_0| + |\mathbf{v}_1||\mathbf{v}_0| \\ &\equiv_{(\text{mod } 2)} \frac{|\mathbf{v}_*|(|\mathbf{v}_*| + 1)}{2} + |\mathbf{v}_*||\mathbf{v}_{01}| + |\mathbf{v}_{01}| + (n + |\mathbf{v}_0|)|\mathbf{v}_1| + |\mathbf{v}_*||\mathbf{v}_0| \\ &\equiv_{(\text{mod } 2)} \frac{|\mathbf{v}_*|(|\mathbf{v}_*| + 1)}{2} + |\mathbf{v}_*||\mathbf{v}_1| + |\mathbf{v}_{01}| + (|\mathbf{v}_*| + |\mathbf{v}_1|)|\mathbf{v}_1| \\ &\equiv_{(\text{mod } 2)} \frac{|\mathbf{v}_*|(|\mathbf{v}_*| + 1)}{2} + |\mathbf{v}_0| \end{aligned}$$

This completes the proof of Theorem 12.

5. More explicit formulas for $m \leq 3$

We derive very concrete expressions using only elementary linear algebra for the case of a small number of hyperplanes. We expect for these formulas to be accessible for a broader range of readers. Furthermore, based on this kind of elementary formulation, we can induce combinatorial identities in Section 6.

5.1. The case of at most two hyperplanes

At first, let us consider only one halfspace, $m = 1$. As we mention before, this case has been revisited in the literature several times. The halfspace,

$$H_1^+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} + r_1 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + r_1 \geq 0\}$$

is an auxiliary plane itself. We get $\mathbf{v}_* = \emptyset, I = \emptyset$ and $\|\mathbf{v}_*\| = |I| = 0, A_\emptyset^\emptyset = 1$ and $A_i^1 = a_i$. The good clipping condition (A) automatically holds and (B) is equivalent to $\prod_{t=1}^n a_t \neq 0$. Applying these terms to Theorem 11 we get a proof for Theorem 1.

Secondly, let us prove Corollary 2. Consider the following two hyperplanes,

$$\begin{aligned} H_1^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} + r_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n + r_1 \geq 0\} \\ H_2^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{b} \cdot \mathbf{x} + r_2 = b_1x_1 + b_2x_2 + \cdots + b_nx_n + r_2 \geq 0\}. \end{aligned}$$

We see that \mathbf{v}_* and I become the empty set or a set of only one element. The former case of the empty set is same as the above one hyperplane case. For the case of $I = [1] = \{1\}$, we put $\mathbf{v}_* = \{*(\mathbf{v})\}$ then $\mathbf{v}_{01} = [n] \setminus \mathbf{v}_*$ and get

$$\prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I \vee m} = \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} a_{*(\mathbf{v})} & b_{*(\mathbf{v})} \\ a_t & b_t \end{vmatrix}.$$

Applying Theorem 12 to these, we get Corollary 2. Here, good clipping conditions are

$$\begin{aligned} \text{(A)} \quad & F^0 \cap H_1 \cap H_2^+ = \emptyset \\ \text{(B)} \quad & \prod_{t=1}^n b_t \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} a_{*(\mathbf{v})} & b_{*(\mathbf{v})} \\ a_t & b_t \end{vmatrix} \neq 0 \quad \text{for } \mathbf{v} \in F^1 \cap H_1 \cap H_2^+. \end{aligned}$$

5.2. The case of three hyperplanes.

Let us consider three halfspaces, $m = 3$. We formulate a concrete form in a similar fashion to the one or two hyperplane cases, in particular, which is used to derive the identity in Appendix C.

Corollary 22. *The volume of the standard unit hypercube $[0, 1]^n$ intersecting the three halfspaces*

$$\begin{aligned} H_1^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} + r_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n + r_1 \geq 0\}, \\ H_2^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{b} \cdot \mathbf{x} + r_2 = b_1x_1 + b_2x_2 + \cdots + b_nx_n + r_2 \geq 0\}, \\ H_3^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c} \cdot \mathbf{x} + r_3 = c_1x_1 + c_2x_2 + \cdots + c_nx_n + r_3 \geq 0\}, \end{aligned}$$

with good clipping assumptions is

$$\begin{aligned} \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+ \cap H_3^+) &= \sum_{\mathbf{v} \in F^0 \cap H_1^+ \cap H_2^+ \cap H_3^+} \frac{(-1)^{|\mathbf{v}|_0} g_3(\mathbf{v})^n}{n! \prod_{t \in [n]} c_t} \\ &- \sum_{\mathbf{v} \in F^1 \cap H_1^+ \cap H_2^+ \cap H_3^+} \frac{(-1)^{|\mathbf{v}|_0} \text{sgn}(a_{*(\mathbf{v})}) a_{*(\mathbf{v})}^{n-1} g_3(\mathbf{v})^n}{n! c_{*(\mathbf{v})} \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} a_{*(\mathbf{v})} & c_{*(\mathbf{v})} \\ a_t & c_t \end{vmatrix}} \\ &- \sum_{\mathbf{v} \in F^1 \cap H_1^+ \cap H_2^+ \cap H_3^+} \frac{(-1)^{|\mathbf{v}|_0} \text{sgn}(b_{*(\mathbf{v})}) b_{*(\mathbf{v})}^{n-1} g_3(\mathbf{v})^n}{n! c_{*(\mathbf{v})} \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} b_{*(\mathbf{v})} & c_{*(\mathbf{v})} \\ b_t & c_t \end{vmatrix}} \end{aligned}$$

$$- \sum_{\substack{\mathbf{v} \in F^2 \cap \\ H_1 \cap H_2 \cap H_3^+}} n! \frac{(-1)^{|\mathbf{v}|_0} \operatorname{sgn} \left(\begin{vmatrix} a_{*1}(\mathbf{v}) & b_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & b_{*2}(\mathbf{v}) \end{vmatrix} \right) \begin{vmatrix} a_{*1}(\mathbf{v}) & b_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & b_{*2}(\mathbf{v}) \end{vmatrix}^{n-1} g_3(\mathbf{v})^n}{\begin{vmatrix} a_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \end{vmatrix} \begin{vmatrix} b_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ b_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \end{vmatrix} \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} a_{*1}(\mathbf{v}) & b_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & b_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \\ a_t & b_t & c_t \end{vmatrix}}.$$

Proof. For each vertex, $|\mathbf{v}_*| = |I| = 0, 1$ or 2 . The former two cases are same as the case of fewer than two hyperplanes. Let us consider the $I = [2]$ cases. Recall $\mathbf{v}_{01} = [n] \setminus \mathbf{v}_*$ and $\mathbf{v}_* = \{*_1(\mathbf{v}), *_2(\mathbf{v})\}$ then

$$\prod_{t \in I} A_{\mathbf{v}_*}^{I \vee m \setminus t} = \begin{vmatrix} a_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \end{vmatrix} \begin{vmatrix} b_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ b_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \end{vmatrix}$$

and

$$\prod_{t \in \mathbf{v}_{01}} A_{\mathbf{v}_* \vee t}^{I \vee m} = \prod_{t \in \mathbf{v}_{01}} \begin{vmatrix} a_{*1}(\mathbf{v}) & b_{*1}(\mathbf{v}) & c_{*1}(\mathbf{v}) \\ a_{*2}(\mathbf{v}) & b_{*2}(\mathbf{v}) & c_{*2}(\mathbf{v}) \\ a_t & b_t & c_t \end{vmatrix}. \quad \square$$

5.3. Examples of calculations

We show two examples of calculations using Corollary 2 and Corollary 22. In particular the following examples have several non-simple vertices. But we can apply our formulas to them because all non-simple vertices lie in the auxiliary hyperplane.

Example 2. Let us calculate the volume of the clipped hypercube $[0, 1]^3$ which intersects the following two halfspaces,

$$\begin{aligned} H_1^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} + \frac{1}{2} = -x_1 + x_2 + \frac{1}{2} \geq 0\}, \\ H_2^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{b} \cdot \mathbf{x} + 3 = -x_1 - 2x_2 - x_3 + 3 \geq 0\}, \end{aligned}$$

Let us find vertices of the clipped hypercube. There are five vertices in $F^0 \cap H_1^+ \cap H_2^+$:

$$\mathbf{v}_1 = (0, 0, 0), \mathbf{v}_2 = (0, 0, 1), \mathbf{v}_3 = (0, 1, 0), \mathbf{v}_4 = (0, 1, 1), \mathbf{v}_5 = (1, 1, 0)$$

and four vertices in $F^1 \cap H_1 \cap H_2^+$:

$$\mathbf{v}_6 = (\frac{1}{2}, 0, 0), \mathbf{v}_7 = (\frac{1}{2}, 0, 1), \mathbf{v}_8 = (1, \frac{1}{2}, 0), \mathbf{v}_9 = (1, \frac{1}{2}, 1).$$

Among those vertices, $\mathbf{v}_4, \mathbf{v}_5$ and \mathbf{v}_9 lie on H_2 and we don't need to worry about these vertices. We can check the good clipping conditions hold. We calculate the values $N_{\mathbf{v}_i}$ for $i = 1, 2, 3, 6, 7, 8$ by Corollary 2. For example, we have

$$N_{\mathbf{v}_6} = - \frac{(-1)^2 \operatorname{sgn}(-1) (-1)^2 g_2(\frac{1}{2}, 0, 0)^3}{3! (-1) \begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix} \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}} = - \frac{(\frac{5}{2})^3}{6 \times 3 \times 1} = - \frac{125}{144}.$$

Therefore we get

$$\begin{aligned}
\text{vol}([0, 1]^3 \cap H_1^+ \cap H_2^+) &= \sum_{\mathbf{v} \in \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}} \frac{(-1)^{|\mathbf{v}|_0} g_2(\mathbf{v})^3}{3! \prod_{t=1}^3 b_t} \\
&\quad - \sum_{\mathbf{v} \in \{\mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8\}} \frac{(-1)^{|\mathbf{v}|_0} a_{*(\mathbf{v})}^3 g_2(\mathbf{v})^3}{3! |a_{*(\mathbf{v})}| b_{*(\mathbf{v})} \prod_{t \in [3] \setminus *(\mathbf{v})} \begin{vmatrix} a_{*(\mathbf{v})} & b_{*(\mathbf{v})} \\ a_t & b_t \end{vmatrix}}, \\
&= \frac{9}{4} - \frac{2}{3} - \frac{1}{12} - \frac{125}{144} + \frac{27}{144} - \frac{1}{36}, \\
&= \frac{19}{24}.
\end{aligned}$$

Remark 3. Note that the polyhedron $[0, 1]^3 \cap H_1^+ \cap H_2^+$ has three non-simple vertices $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_9$ but all these are degenerate so we can apply the formula. Therefore if one changes the roles of the two halfspaces then one cannot apply the formula, because there are non-degenerate non-simple vertices and it would violate the good clipping conditions.

Example 3. Let us calculate the volume of the region of $[0, 1]^3$ that intersects the three halfspaces,

$$\begin{aligned}
H_1^+ &= \{x \mid -x_1 + x_2 + \frac{1}{2} \geq 0\}, \\
H_2^+ &= \{x \mid x_3 - \frac{1}{2} \geq 0\}, \\
H_3^+ &= \{x \mid -x_1 - 2x_2 - x_3 + 3 \geq 0\}.
\end{aligned}$$

The three halfspaces satisfy the good clipping conditions and we can apply Corollary 22. Let us find vertices according to each $I \subset [3 - 1]$, i.e. $I = \emptyset, \{1\}, \{2\}$ and $\{1, 2\}$

$$\begin{aligned}
F^0 \cap H_1^+ \cap H_2^+ \cap H_3^+ &: \mathbf{v}_1 = (0, 0, 1), \mathbf{v}_2 = (0, 1, 1), \\
F^1 \cap H_1 \cap H_2^+ \cap H_3^+ &: \mathbf{v}_3 = (\frac{1}{2}, 0, 1), \mathbf{v}_4 = (1, \frac{1}{2}, 1), \\
F^1 \cap H_1^+ \cap H_2 \cap H_3^+ &: \mathbf{v}_5 = (0, 0, \frac{1}{2}), \mathbf{v}_6 = (0, 1, \frac{1}{2}), \\
F^2 \cap H_1 \cap H_2 \cap H_3^+ &: \mathbf{v}_7 = (\frac{1}{2}, 0, \frac{1}{2}), \mathbf{v}_8 = (1, \frac{1}{2}, \frac{1}{2}).
\end{aligned}$$

Let us check that \mathbf{v}_2 and \mathbf{v}_4 lie on H_3 and these vertices are degenerate vertices. Note that there are two more degenerate vertices $\mathbf{v}_9 = (1, \frac{3}{4}, \frac{1}{2})$ and $\mathbf{v}_{10} = (\frac{1}{2}, 1, \frac{1}{2})$ which are excluded from the summation automatically so we don't need to care. In summary, the polyhedron $[0, 1]^3 \cap H_1^+ \cap H_2^+ \cap H_3^+$ has ten vertices with two non-simple vertices \mathbf{v}_2 and \mathbf{v}_4 among them. After applying Corollary 22 to these, we obtain

$$N_{\mathbf{v}_1} = -\frac{2}{3}, N_{\mathbf{v}_3} = \frac{3}{16}, N_{\mathbf{v}_5} = \frac{125}{96}, N_{\mathbf{v}_6} = -\frac{1}{96}, N_{\mathbf{v}_7} = -\frac{4}{9}, \text{ and } N_{\mathbf{v}_8} = -\frac{1}{288}.$$

Therefore we obtain the volume as follows,

$$\begin{aligned}
\text{vol}([0, 1]^3 \cap H_1^+ \cap H_2^+ \cap H_3^+) &= \sum_{\mathbf{v}=\mathbf{v}_1} N_{\mathbf{v}} + \sum_{\mathbf{v}=\mathbf{v}_3} N_{\mathbf{v}} + \sum_{\mathbf{v} \in \{\mathbf{v}_5, \mathbf{v}_6\}} N_{\mathbf{v}} + \sum_{\mathbf{v} \in \{\mathbf{v}_7, \mathbf{v}_8\}} N_{\mathbf{v}}, \\
&= -\frac{2}{3} + \frac{3}{16} + \frac{125}{96} - \frac{1}{96} - \frac{4}{9} - \frac{1}{288}, \\
&= \frac{35}{96}.
\end{aligned}$$

6. Combinatorial identities from clipping hypercubes

6.1. From polytopes to identities

Let us consider a general methodology producing from a polytope volume a combinatorial identity. This is a simple observation that the resulting volume is independent of the choice of an auxiliary plane. Recall the volume expression of Theorem 7 and theorems in Section 4.3 and let us assume that we already know the volume of a clipped hypercube $P = [0, 1]^n \cap H_1 \cap \dots \cap H_{m-1}$. Let us cut P into two pieces

$$P_+ = P \cap H_m^+ \quad \text{and} \quad P_- = P \cap H_m^-$$

one more time by the auxiliary hyperplane

$$H_m = \{a_1 x_1 + \dots + a_n x_n + y = 0\}.$$

No matter how we take H_m , the union of two pieces should be P and

$$\text{vol}(P_+) + \text{vol}(P_-) = \text{vol}(P).$$

Therefore the known volume is constant and expressed in terms of the free variables a_1, a_2, \dots, a_n and y , which produces an algebraic identity.

Remark 4. Note that there might be several constraints on the indeterminate variables, for examples, good clipping conditions for using the volume formula. But these constraints can be removed by continuity as long as it is well-defined.

Remark 5. When we look at volume formulas, we can see the volume expression is homogeneous for a_1, a_2, \dots, a_n because it is composed of homogeneous polynomials which are determinants of matrices with one column vector of indeterminate $\mathbf{a} = (a_1, a_2, \dots, a_n)$.

Let us see the most simple case which is a direct consequence of Theorem 1.

Corollary 23. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Then

$$\sum_{\mathbf{v} \in F^0} (-1)^{|\mathbf{v}_0|} (\mathbf{a} \cdot \mathbf{v} + y)^n = n! a_1 a_2 \dots a_n$$

Proof. Since

$$\text{vol}([0, 1]^n \cap H^-) = \sum_{\mathbf{v} \in F^0 \cap H^-} \frac{(-1)^{|\mathbf{v}_0|} g(\mathbf{v})^n}{n! \prod_{t=1}^n a_t}$$

and

$$\text{vol}([0, 1]^n) = \text{vol}([0, 1]^n \cap H^+) + \text{vol}([0, 1]^n \cap H^-),$$

so we have

$$1 = \sum_{\mathbf{v} \in F^0} \frac{(-1)^{|\mathbf{v}_0|} g(\mathbf{v})^n}{n! \prod_{t=1}^n a_t}.$$

Finally, we get

$$\sum_{\mathbf{v} \in F^0} (-1)^{|\mathbf{v}_0|} (\mathbf{a} \cdot \mathbf{v} + y)^n = n! \prod_{t=1}^n a_t. \quad \square$$

Let us consider the summation over $\mathbf{v} \in F^0 = \{0, 1\}^n$. We can replace this summation by $1 \leq a_{t_1}, \dots, a_{t_i} \leq n$ as regarding $\mathbf{v}_1 = \{a_{t_1}, \dots, a_{t_i}\}$ and hence prove Theorem 3. Note that a_i should be nonzero when applying the volume formula but the resulting identity has no such constraint by continuity, as we remarked above.

Essentially, whenever we take a polytope, we can find a corresponding combinatorial identity using an exact volume formula. So we can expect this kind of

$$\{ \text{polytopes} \} \longrightarrow \{ \text{combinatorial identities} \}$$

correspondence has a structural property. At this stage, it seems to be somewhat vague to investigate the resulting identities from general convex polytopes. Nevertheless we can figure out several cases. We give the case of a clipped hypercube by a symmetric hyperplane with full generality in the next section, which produces the very interesting identity in Theorem 6. Moreover, we treat several examples of resulting identities in the [Appendix](#).

6.2. Symmetric arrangements of a hyperplane

When we see the identity of Theorem 3, we can observe that this is a symmetric function of the n -variables a_1, \dots, a_n . This property comes from the fact that the polytope under consideration is symmetric, i.e. we took a symmetric arrangement of hyperplanes, where the term *symmetric* means that hyperplanes except the auxiliary hyperplane are invariant under exchange of coordinate axes of \mathbb{R}^n .

Probably the second easiest example of a symmetric arrangement is

$$H_1 = \{\mathbf{x} \in \mathbb{R}^n \mid -x_1 - x_2 - \dots - x_n + 1 \geq 0\}.$$

We use the auxiliary plane

$$H_2 = \{\mathbf{x} \in \mathbb{R}^n \mid a_1 x_1 + \dots + a_n x_n + y \geq 0\}$$

in the formula of the $m = 2$ case and obtain the identity of Theorem 4. We remark that the identity of Theorem 4 is a direct consequence of Proposition 1 in [23].

Note that H_1 and H_2 violate good clipping conditions. So we use the ϵ -perturbation method in Section 4.2 when applying the volume formula.

Proof of Theorem 4. For sufficiently small $\epsilon > 0$, let us consider

$$H_1^+ = \{\mathbf{x} \mid -x_1 - x_2 - \dots - x_n + 1 - \epsilon \geq 0\},$$

then it satisfies good clipping conditions. We have

$$\begin{aligned} \text{vol}([0, 1]^n \cap H_1^+) &= \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+) + \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^-) \\ &= \sum_{\mathbf{v} \in F^0 \cap H_1^+ \cap H_2^+} - \sum_{\mathbf{v} \in F^1 \cap H_1 \cap H_2^+} + \sum_{\mathbf{v} \in F^0 \cap H_1^+ \cap H_2^-} - \sum_{\mathbf{v} \in F^1 \cap H_1 \cap H_2^-} \\ &= \sum_{\mathbf{v} \in F^0 \cap H_1^+} \frac{(-1)^{|\mathbf{v}_0|} g_2(\mathbf{v})^n}{n! \prod_{t=1}^n a_t} \\ &\quad - \sum_{\mathbf{v} \in F^1 \cap H_1} \frac{(-1)^{|\mathbf{v}_0|} (-1)^n g_2(\mathbf{v})^n}{n! a_{*}(\mathbf{v}) \prod_{t=1, t \neq *(\mathbf{v})}^n \left| \begin{array}{cc} -1 & a_{*}(\mathbf{v}) \\ -1 & a_t \end{array} \right|}. \end{aligned}$$

Then there are $n + 1$ vertices of

$$\begin{aligned} \mathbf{v} &= (0, 0, \dots, 0) && \in F^0 \cap H_1^+ \\ \mathbf{v}_i &= (1 - \epsilon)\mathbf{e}_i && \in F^1 \cap H_1 \quad \text{for } i \in [n]. \end{aligned}$$

Hence we obtain

$$\frac{(1 - \epsilon)^n}{n!} = \frac{(-1)^n y^n}{n! a_1 a_2 \cdots a_n} - \sum_{i=1}^n \frac{(-1)^{n-1} (-1)^n (a_i(1 - \epsilon) + y)^n}{n! a_i \prod_{t=1, t \neq i}^n (a_i - a_t)}.$$

By taking $\epsilon \rightarrow 0$ and simplifying, we conclude the result. \square

We next consider the following hyperplane,

$$H_1 = \{\mathbf{x} \in \mathbb{R}^n \mid -x_1 - x_2 - \cdots - x_n + 2 \geq 0\}.$$

By the same ϵ -perturbation taking $2 - \epsilon$ instead of 2, we can get the following identity,

$$\begin{aligned} \frac{y^n}{a_1 a_2 \cdots a_n} - \sum_{i=1}^n \frac{(y + a_i)^n}{a_1 a_2 \cdots a_n} + \\ \sum_{1 \leq t_1 < t_2 \leq n} \sum_{i=1}^2 \frac{(y + a_{t_1} + a_{t_2})^n}{a_{t_i}(a_1 - a_{t_i})(a_2 - a_{t_i}) \cdots (a_n - a_{t_i})} = (-1)^n (2^n - n). \end{aligned}$$

We consider all possible symmetric arrangements of only one hyperplane. Then all linear coefficients of H_1 should be the same. So it is reasonable to think about

$$H_l = \{\mathbf{x} \in \mathbb{R}^n \mid -x_1 - x_2 - \cdots - x_n + l \geq 0\} \quad \text{for } l \in [n].$$

This gives the following theorem which is nothing but a different form of Theorem 6.

Theorem 24. *For an integer $l \in [n]$ and non-zero distinct real numbers a_1, a_2, \dots, a_n and $y \in \mathbb{R}$,*

$$\begin{aligned} \frac{y^n}{a_1 a_2 \cdots a_n} + \sum_{i=1}^{l-1} \sum_{1 \leq t_1 < t_2 < \cdots < t_i \leq n} \frac{(-1)^i (y + a_{t_1} + a_{t_2} + \cdots + a_{t_i})^n}{a_1 a_2 \cdots a_n} \\ + (-1)^l \sum_{1 \leq t_1 < \cdots < t_l \leq n} \sum_{i=1}^l \frac{(y + a_{t_1} + \cdots + a_{t_l})^n}{a_{t_i}(a_1 - a_{t_i})(a_2 - a_{t_i}) \cdots (a_n - a_{t_i})} \\ = \sum_{i=0}^{l-1} (-1)^{n-i} \binom{n}{i} (l-i)^n \end{aligned}$$

or equivalently

$$\begin{aligned}
& \sum_{i=1}^{l-1} \sum_{1 \leq t_1 < t_2 < \dots < t_i \leq n} (-1)^i (a_{t_1} + a_{t_2} + \dots + a_{t_i})^k \\
& + (-1)^l \sum_{1 \leq t_1 < \dots < t_l \leq n} \sum_{i=1}^l \left(\prod_{j=1, j \neq t_i}^n \frac{a_j}{a_j - a_{t_i}} \right) (a_{t_1} + \dots + a_{t_l})^k \\
& = \begin{cases} -1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, \dots, n-1 \\ \prod_{i=1}^n a_i \sum_{i=0}^{l-1} (-1)^{n-i} \binom{n}{i} (l-i)^n & \text{if } k = n \end{cases}
\end{aligned}$$

or, using set-notation with $A = \{a_1, a_2, \dots, a_n\}$,

$$\begin{aligned}
& \sum_{I \subset A}^{|I| < l} (-1)^{|I|} (y + \|I\|)^n + \sum_{I \subset A}^{|I| = l} (-1)^l (y + \|I\|)^n \left(\sum_{a \in I} \prod_{b \in A \setminus a} \frac{b}{b-a} \right) \\
& = A! \sum_{i=0}^{l-1} (-1)^{n-i} \binom{n}{i} (l-i)^n.
\end{aligned}$$

Be cautious that the difference between Theorem 6 and the second form above occurs for the case $k = 0$ which comes from $\|\emptyset\|^0 = 0^0 = 1$.

Proof. We use ϵ -perturbation replacing l by $l - \epsilon$ for H_1 , then

$$\begin{aligned}
\text{vol } ([0, 1]^n \cap H_1^+) &= \sum_{\mathbf{v} \in F^0 \cap H_1^+} \frac{(-1)^{|\mathbf{v}_0|} (\mathbf{a} \cdot \mathbf{v} + y)^n}{n! a_1 a_2 \dots a_n} \\
&- \sum_{\mathbf{v} \in F^1 \cap H_1} \frac{(-1)^{n-l} (-1)^n (\mathbf{a} \cdot \mathbf{v} + y)^n}{n! a_*(\mathbf{v}) \prod_{t=1, t \neq *(\mathbf{v})} (a_*(\mathbf{v}) - a_t)}.
\end{aligned}$$

For $F^0 \cap H_1^+$, there are a total of $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{l-1}$ vertices, i.e. there are l families of vertices with respect to the sum of coordinate values of \mathbf{v} .

Also, for $F^1 \cap H_1$, there are $l \binom{n}{l}$ vertices whose coordinate values are $l-1$ one's, unique $1 - \epsilon$, and $n-l$ zero's. Hence we obtain

$$\begin{aligned}
& \sum_{\mathbf{v} \in F^0 \cap H_1^+} \frac{(-1)^{|\mathbf{v}_0|} (\mathbf{a} \cdot \mathbf{v} + y)^n}{n! a_1 a_2 \dots a_n} \\
& = \frac{(-1)^n y^n}{n! a_1 a_2 \dots a_n} + \sum_{i=1}^{l-1} \sum_{1 \leq t_1 < \dots < t_i \leq n} \frac{(-1)^{n-i} (y + a_{t_1} + \dots + a_{t_i})^n}{n! a_1 a_2 \dots a_n}
\end{aligned}$$

and

$$\sum_{\mathbf{v} \in F^1 \cap H_1} \frac{(-1)^{l+1} (\mathbf{a} \cdot \mathbf{v} + y)^n}{n! a_*(\mathbf{v}) \prod_{t=1, t \neq *(\mathbf{v})} (a_*(\mathbf{v}) - a_t)}$$

$$= (-1)^{l+1} \sum_{1 \leq t_1 < \dots < t_l \leq n} \sum_{i=1}^l \frac{(y + a_{t_1} + \dots + a_{t_i}(1 - \epsilon) + \dots + a_{t_l})^n}{n! a_{t_i}(a_{t_i} - a_1)(a_{t_i} - a_2) \dots (a_{t_i} - a_n)}.$$

We compute the volume of clipped hypercube using Theorem 1

$$\begin{aligned} \text{vol } ([0, 1]^n \cap H_1^+) &= \frac{(-1)^n (l - \epsilon)^n}{n! (-1)^n} + \frac{(-1)^{n-1} \binom{n}{1} (l - 1 - \epsilon)^n}{n! (-1)^n} + \dots \\ &\quad + \frac{(-1)^{n-(l-1)} \binom{n}{l-1} (l - (l-1) - \epsilon)^n}{n! (-1)^n} \\ &= \sum_{i=0}^{l-1} (-1)^i \binom{n}{i} \frac{(l - i - \epsilon)^n}{n!}. \end{aligned}$$

By taking $\epsilon \rightarrow 0$, we conclude the result. \square

Remark 6. If we take l to be a non-integer real number, we get a slightly different identity, obtained by rescaling variables from the result of Theorem 24. If the vertex configuration is preserved under changing hyperplanes, the resulting identity is essentially the same as the previous one.

Appendix. Several clipped hypercube identities

For simplicity, we do not use m for the number of hyperplanes in the appendix section and use \mathbf{o}_n instead of $(0, 0, \dots, 0)$ in \mathbb{R}^n .

A. Symmetric truncated hypercube

Let us consider $n + 1$ hyperplanes

$$\begin{aligned} H_1 &= \{\mathbf{x} \mid -x_1 + x_2 + \dots + x_n + 1 - d = 0\} \\ H_2 &= \{\mathbf{x} \mid x_1 - x_2 + \dots + x_n + 1 - d = 0\} \\ &\vdots \\ H_n &= \{\mathbf{x} \mid x_1 + x_2 + \dots - x_n + 1 - d = 0\} \\ H_{n+1} &= \{\mathbf{x} \mid a_1 x_1 + a_2 x_2 + \dots + a_n x_n + y = 0\}, \end{aligned}$$

Then the volume is the following.

$$\begin{aligned} \text{vol } ([0, 1]^n \cap H_1^+ \cap \dots \cap H_n^+) &= \text{vol } ([0, 1]^n \cap H_1^+ \cap \dots \cap H_n^+ \cap H_{n+1}^+) + \text{vol } ([0, 1]^n \cap H_1^+ \cap \dots \cap H_n^+ \cap H_{n+1}^-) \\ &= 1 - n \times \frac{d^n}{n!} \end{aligned}$$

Note that these hyperplanes do not intersect each other in $[0, 1]^n$ under the condition $0 < d < 1$. Corollary 2 essentially suffices to compute the volume.

We can check that there are three kinds of vertices,

$$\begin{aligned} |I| = 0 : & \quad F^0 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}, \\ |I| = 1 : & \quad (1-d)\mathbf{e}_i \quad \text{for } i = 1, 2, \dots, n, \\ & \quad \mathbf{e}_i + d\mathbf{e}_j \quad \text{for } 1 \leq i \neq j \leq n. \end{aligned}$$

The resulting identity is

$$\begin{aligned} & \sum_{i=1}^n \frac{(y+a_i)^n}{a_1 a_2 \cdots a_n} - \sum_{i=1}^n \frac{(y+a_i(1-d))^n}{a_i \prod_{j=1, j \neq i}^n (a_j + a_i)} \\ & - \sum_{1 \leq i \neq j \leq n} \frac{(y+a_i+a_j d)^n}{a_j(a_i+a_j) \prod_{t=1, t \neq i, j}^n (a_t - a_j)} = (-1)^{n+1} n d^n \end{aligned}$$

B. Hyperprism : n -simplex $\times [0, 1]^m$

Let us consider the following two hyperplanes

$$\begin{aligned} H_1 &= \{\mathbf{x} \mid -x_1 - x_2 - \cdots - x_n + 1 - \epsilon = 0\} \\ H_2 &= \{\mathbf{x} \mid a_1 x_1 + \cdots + a_n x_n + b_1 x_{n+1} + \cdots + b_m x_{n+m} + y = 0\}, \end{aligned}$$

The resulting volume taking $\epsilon \rightarrow 0$ is the following.

$$\begin{aligned} \text{vol}([0, 1]^{n+m} \cap H_1^+) &= \text{vol}([0, 1]^{n+m} \cap H_1^+ \cap H_2^+) + \text{vol}([0, 1]^{n+m} \cap H_1^+ \cap H_2^-) \\ &= \frac{1}{n!}. \end{aligned}$$

We can check that there are several kinds of vertices

$$\begin{aligned} |I| = 0 : & \quad \mathbf{o}_{n+m}, \\ & \quad \mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \dots, \mathbf{e}_{n+m}, \\ & \quad \mathbf{e}_{n+1} + \mathbf{e}_{n+2}, \mathbf{e}_{n+1} + \mathbf{e}_{n+3}, \dots, \mathbf{e}_{n+m-1} + \mathbf{e}_{n+m} \\ & \quad \vdots \\ & \quad \mathbf{e}_{n+1} + \mathbf{e}_{n+2} + \cdots + \mathbf{e}_{n+m}, \end{aligned}$$

and for $i \in [n]$,

$$\begin{aligned} |I| = 1 : & \quad (1-\epsilon)\mathbf{e}_i + \mathbf{o}_{n+m}, \\ & \quad (1-\epsilon)\mathbf{e}_i + \{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+m}\}, \\ & \quad (1-\epsilon)\mathbf{e}_i + \{\mathbf{e}_{n+1} + \mathbf{e}_{n+2}, \mathbf{e}_{n+1} + \mathbf{e}_{n+3}, \dots, \mathbf{e}_{n+m-1} + \mathbf{e}_{n+m}\} \\ & \quad \vdots \\ & \quad (1-\epsilon)\mathbf{e}_i + \mathbf{e}_{n+1} + \mathbf{e}_{n+2} + \cdots + \mathbf{e}_{n+m}. \end{aligned}$$

The resulting identity is

$$\begin{aligned}
& \sum_{I \subset \{b_1, \dots, b_m\}} (-1)^{|I|} (y + \|I\|)^{n+m} \\
& + \sum_{i=1}^n \sum_{I \subset \{b_1, \dots, b_m\}} (-1)^{|I|+1} \left(\prod_{j=1, j \neq i}^n \frac{a_j}{a_j - a_i} \right) (y + a_i + \|I\|)^{n+m} \\
& = (-1)^{n+m} \frac{(n+m)!}{n!} a_1 \cdots a_n b_1 \cdots b_m.
\end{aligned}$$

C. Isosceles n -simplex

Let us consider the following three hyperplanes

$$\begin{aligned}
H_1 &= \{\mathbf{x} \mid -x_1 - x_2 - \cdots - x_n + 1 - \epsilon = 0\} \\
H_2 &= \{\mathbf{x} \mid x_1 - x_2 - \cdots - x_n - \epsilon = 0\} \\
H_3 &= \{\mathbf{x} \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + y = 0\}.
\end{aligned}$$

The resulting volume taking $\epsilon \rightarrow 0$ is the following.

$$\begin{aligned}
& \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+) \\
& = \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+ \cap H_3^+) + \text{vol}([0, 1]^n \cap H_1^+ \cap H_2^+ \cap H_3^-) \\
& = \frac{1}{n! 2^{n-1}}.
\end{aligned}$$

We can check that there are two kinds of vertices

$$\begin{aligned}
|I| = 1 &: \quad \epsilon \mathbf{e}_1, (1 - \epsilon) \mathbf{e}_1, \\
|I| = 2 &: \quad \frac{1}{2} \mathbf{e}_1 + \left(\frac{1}{2} - \epsilon\right) \mathbf{e}_i \quad \text{for } 2 \leq i \leq n.
\end{aligned}$$

This case needs Corollary 22 for the three hyperplane case.

The resulting identity is

$$\begin{aligned}
& \frac{y^n}{a_1(a_2 + a_1)(a_3 + a_1) \cdots (a_n + a_1)} - \frac{(y + a_1)^n}{a_1(a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1)} \\
& - 2 \sum_{i=2}^n \frac{(y + \frac{a_1}{2} + \frac{a_i}{2})^n}{(a_1 + a_i)(a_1 - a_i)(a_2 - a_i) \cdots (a_n - a_i)} = (-1)^n 2^{1-n}.
\end{aligned}$$

D. Trapezoidal polytope

Let us consider the following two hyperplanes

$$\begin{aligned}
H_1 &= \{\mathbf{x} \mid -\frac{x_1}{2} - \frac{x_2}{2} - \cdots - \frac{x_n}{2} - x_{n+1} - x_{n+2} - \cdots - x_{n+m} + 1 - \epsilon = 0\}, \\
H_2 &= \{\mathbf{x} \mid a_1 x_1 + \cdots + a_n x_n + b_1 x_{n+1} + \cdots + b_m x_{n+m} + y = 0\}.
\end{aligned}$$

The resulting volume taking $\epsilon \rightarrow 0$ is the following.

$$\text{vol}([0, 1]^{n+m} \cap H_1^+) = \text{vol}([0, 1]^{n+m} \cap H_1^+ \cap H_2^+) + \text{vol}([0, 1]^{n+m} \cap H_1^+ \cap H_2^-)$$

$$= \frac{2^n - n2^{-m}}{(n+m)!}.$$

We can check that there are four kinds of vertices:

$$\begin{aligned} |I| = 0 : & \quad \mathbf{o}_{n+m}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \\ |I| = 1 : & \quad (1-\epsilon)\mathbf{e}_{n+i} \quad \text{for } 1 \leq i \leq m, \\ & \quad \mathbf{e}_i + \left(\frac{1}{2} - \epsilon\right)\mathbf{e}_{n+j} \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq m, \\ & \quad \mathbf{e}_i + (1-2\epsilon)\mathbf{e}_j \quad \text{for } 1 \leq i \neq j \leq n. \end{aligned}$$

The resulting identity is

$$\begin{aligned} & \frac{(-1)^{n+m}y^{n+m}}{a_1 \cdots a_n b_1 \cdots b_m} + \sum_{i=1}^n \frac{(-1)^{n+m-1}(y+a_i)^{n+m}}{a_1 \cdots a_n b_1 \cdots b_m} \\ & + \sum_{j=1}^m \frac{(y+b_j)^{n+m}}{b_j \prod_{s=1}^n \left(\frac{b_j}{2} - a_s\right) \prod_{t=1, t \neq j}^m (b_j - b_t)} \\ & - \sum_{i=1}^n \sum_{j=1}^m \frac{(y+a_i + \frac{b_j}{2})^{n+m}}{b_j \prod_{s=1}^n \left(\frac{b_j}{2} - a_s\right) \prod_{t=1, t \neq j}^m (b_j - b_t)} \\ & - \sum_{1 \leq i \neq j \leq n} \frac{(y+a_i + a_j)^{n+m}}{2^m a_i \prod_{s=1, s \neq i}^n (a_i - a_s) \prod_{t=1}^m (a_i - \frac{b_t}{2})} \\ & = 2^n - n2^{-m}. \end{aligned}$$

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